MathMagics

Divisibility, Binary, and Magic Squares

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Math Circle Handout

“Mathematicians are the world’s greatest magicians — we make patterns appear from nowhere.”

Warm-Up Magic — The 1089 Trick

1. Think of any 3-digit number where the first and last digits are different.
2. Reverse it and subtract the smaller from the larger.
3. Reverse that result and add them together.

Magic: You always get the same number! Try it below:

Example 1: ___ → ___ → ___ → ___   Example 2: ___ → ___ → ___ → ___

Think: Why might this always work? Hint: express the 3-digit number as $100a + 10b + c$?

Divisibility Magics

Divisible by 9: Add the digits. If the sum is a multiple of 9, so is the number!
Example: $45909 \Rightarrow 4+5+9+0+9 = 27 \rightarrow$ divisible by 9.
Divisible by 11: Subtract and add alternating digits.
Example: $3146 \Rightarrow 3-1+4-6 = 0 \rightarrow$ divisible by 11.
Divisible by 7: Double the last digit, subtract it from the rest, repeat if needed. **Example:** $203 \Rightarrow 20 - 2\times 3 = 14 \rightarrow$ divisible by 7!

Try your own!

1. 4275 Divisible by 9? ___
2. 264 Divisible by 11? ___
3. 399 Divisible by 7? ___

Challenge: Invent your own divisibility rule for a new number like 13 or 17. Does it work?

Guess the Missing Digit!

The Trick (Performance Script)

1. Ask a volunteer to think of any three-digit number. (Example: $n = 753$.)
2. Tell them: "Now add up the digits of your number, and subtract this sum from your original number." (For example: $753 - (7 + 5 + 3) = 738$.)
3. Ask them to tell you the sum of *any two* of the three digits in their new number.
4. Pause dramatically... and then reveal the *third digit*¡

The Secret. When you subtract the sum of a number's digits from the number itself, the result is always a multiple of 9. That means the digits of the result always add up to either 9, 18, or 27 — in other words, the digits of the final number always sum to a multiple of 9.
So if the volunteer gives you the sum of any two digits, you can instantly find the missing one:

$$\text{Missing digit} = 9 - (\text{sum of the two given digits}) \quad (\text{or } 18 - \text{sum if needed}).$$

Example.

$$\begin{aligned} \text{Volunteer picks: } & 524 \\ \text{Subtract sum of digits: } & 524 - (5 + 2 + 4) = 513 \\ \text{Digits: } & 5, 1, 3 \\ \text{They say: } & 5 + 3 = 8 \\ \text{You say: } & 9 - 8 = 1 \text{ — the missing digit!} \end{aligned}$$

Performance Tips.

- Speak slowly, as if you are "computing" the answer in your head — it adds suspense.
- If the audience asks why the trick works, smile and say, "Because numbers always tell the truth... eventually."
- Try it with 4-digit numbers too — can you still predict the missing one?

Think:

- Why must the result always be divisible by 9?
- What does this have to do with the "divisibility by 9" rule you learned earlier?
- Could you design a similar trick that works with a different base (say, base 8 or base 12)?

Binary Mind Reading

Imagine you have 4 cards, each contains some numbers from 1 to 15. Ask a friend to think of a number between 1 and 15 and tell you which cards their number appears on. You add up the first numbers on those cards — and boom, you know their number!
Magic? Nope — it's *binary.* Each card represents a binary digit:

$$1,\ 2,\ 4,\ 8$$

Try it! If the secret number appears on cards 1, 2, and 5 → what is it?
Think: Why does this work? Could you design a similar game using base 3 (with powers of 3: 1, 3, 9, 27, . . .)?

Card MathMagic — The 27-Card Trick

Effect:

1. Take any 27 cards from a deck.
2. Ask a volunteer to secretly choose *one card* and also a *number between 1 and 27.*
3. Deal the cards face up into **3 columns of 9 cards each** (row by row).
4. Ask which column their card is in.
5. Pick up the piles in some order (this is the magical part!).
6. Repeat this dealing-and-stacking process **two more times**.
7. Finally, ask the volunteer to count down to their chosen number —
   the card at that position is their card!

Question: Why it works? Think!
Try it:

- Perform it once slowly with a friend. Count carefully!
- What happens if you start with only 9 cards (3 columns of 3)?
- What if you try 81 cards (3 columns of 27)? How many rounds do you think you'll need now?
- Can you find a general rule for how many rounds are needed when you start with 3^k cards?

Think: Suppose the volunteer chose the number 22. How should you decide the order in which to stack the columns each round to make sure the trick succeeds?

Bonus challenge: Can you invent your own card trick that always reveals the chosen card after a fixed number of rounds? Do you have any other creative way of exploring the chosen card?

The Grid Trick

Here's a mysterious grid:

8	3	4
1	5	9
6	7	2

Add any row, column, or diagonal — what do you get?
Challenge:

- Can you make your own 3×3 grid with all sums equal?
- What if you only use even numbers? Odd numbers? Prime numbers?

Magic Squares — The Birthday Trick

A **magic square** is a grid of numbers where each row, column, and diagonal has the same sum.
Try this 4×4 version:

Choose a target number (for example, your birthday date!) and fill the grid so that every line sums to it.
Questions:

- Can you make a 3×3 magic square with all odd numbers?
- Is there a 2×2 magic square? Why or why not?
- Can you make a "magic rectangle" (3×4 grid) where rows have equal sums but columns don't?

Magic Squares — Your Number, Your Trick

The Trick (Performance Script)

1. Invite someone to name *any* whole number n.
2. Say: "Great! I'll build a 4×4 magic square whose magic sum is exactly n."
3. Fill the grid using the recipe below (it takes just a few seconds).
4. Show that every row, column, and both diagonals add up to n.
5. Bonus wow: each 2×2 corner block also adds up to n!

The Secret Recipe

Given the chosen number n, write this 4×4 square:

$n - 20$	1	12	7
11	8	$n - 21$	2
5	10	3	$n - 18$
4	$n - 19$	6	9

Why It Works (Quick Proof)

Each row, column, diagonal, and even the four 2×2 corner blocks all sum to n. Try verifying a few lines yourself!

Example: If the audience says $n = 34$, the square becomes

14	1	12	7
11	8	13	2
5	10	3	16
4	15	6	9

Every row, column, and diagonal sums to 34.

Performance Tips

- Talk as you write: "Top left is $n - 20$, then $1, 12, 7 \ldots$"
- Let the audience pick a random row or column to test first.
- Reveal the 2×2 block property last — it's the best surprise.

Think: Can you find another 4×4 recipe that works for every n? What makes this particular arrangement so powerful?

Practice: Build the square for these values and verify the sums:

$$n \in \{25,\ 31,\ 40,\ 57,\ 100\}.$$

A Short History and Theory of Magic Squares

A Little History. Magic squares have fascinated mathematicians, mystics, and artists for more than two thousand years. The oldest known example is the *Lo Shu Square* from ancient China (about 2200 BCE):

4	9	2
3	5	7
8	1	6

Every row, column, and diagonal adds up to 15. The legend tells of a turtle emerging from the Yellow River with this pattern on its shell — it became a symbol of harmony and balance. Similar constructions later appeared in India, the Islamic world, and Renaissance Europe. Even the artist Albrecht Dürer hid a 4×4 magic square in his engraving *Melencolia I* (1514), where the bottom row secretly contains the digits "1514," the year of the artwork!

How Mathematicians Build Them. For odd orders like 3×3, 5×5, 7×7, a classical method called the *Siamese method* (or de la Loubère's method) works:

1. Start with 1 in the middle of the top row.
2. Move one step up and one step right for each new number.
3. If that spot is already filled, move one step straight down instead.

This simple rule produces a perfect 3×3 magic square using numbers 1–9 with constant sum 15. For even orders like 4×4, there are other "interlacing" and "complementary pair" constructions — one of which powers your performance version.

Why It Works (The Algebra Behind the Magic). At its heart, a magic square is a clever balancing act of linear equations. If a_{ij} is the entry in row i, column j, then requiring all rows, columns, and diagonals to have the same sum means:

$$a_{i1} + a_{i2} + a_{i3} + a_{i4} = \text{constant}, \quad a_{1j} + a_{2j} + a_{3j} + a_{4j} = \text{constant}, \quad \text{etc.}$$

Your "universal" 4×4 recipe works because each a_{ij} is chosen so that every line contains one copy of each "offset" relative to n. In algebraic terms, all rows, columns, and diagonals contain numbers that sum to the same linear combination of n and constants, giving total sum n.

Can We Do the Same for a 3×3 Square? For a fixed pattern (like Lo Shu), we can multiply all entries by a constant k and add another constant c to get a new magic sum:

$$M'_{ij} = kM_{ij} + c,$$

so the new magic sum becomes $k \times 15 + 3c$. However, there is no single 3×3 *integer* pattern that can adjust to *any* number n with only integer arithmetic like your 4×4 formula does — the symmetry of 3×3 is too tight to allow all degrees of freedom independently. In short: you can scale and shift the Lo Shu square to get many magic sums, but not an "any-number-you-like" version with the same simplicity.

Think:

- Why might the 4×4 version have enough flexibility to "absorb" the chosen n?
- What happens if we try to design a similar 5×5 universal recipe?
- Could there be an algebraic reason (degrees of freedom) explaining which sizes allow universal patterns?

Group Challenge — Think Like a MathMagician!

Work together on one or more of these puzzles:

1. Why does the "sum of digits" trick depend on the number 10? What if our base system were 12?
2. What happens if you apply the 1089 trick to a 4-digit number?
3. How many numbers can you represent using 7 binary cards?
4. What's the smallest number of cards you need to represent 100?
5. How many different 3×3 magic squares exist (up to rotation/reflection)?
6. Can you create your own math magic using patterns, remainders, or symmetry?

Brainstorm ideas and test your own tricks!

"The real magic is not the trick — it's discovering the math behind it."