A Brief Introduction to Combinatorial (Positional) Games

Asaf Ferber*

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1 Introduction

How can one systematically analyze a game such as Chess, Go, Checkers, Tic-Tac-Toe, or Hex? A naive attempt that explores all future possibilities quickly runs into an astronomical number of positions. Indeed, the "state space" of each move tree typically grows so large that brute force analysis by a computer (or a person) is hopeless, even though the game is finite and theoretically decidable.

Traditional *Game Theory* focuses mostly on *incomplete information* or *stochastic* games, such as Poker, where chance or hidden cards play a key role. In contrast, *Combinatorial Game Theory* (sometimes also called *Positional Game Theory*) focuses on *perfect information* games with no randomness in the rules. Chess, Go, and Hex are classic examples: both players know the entire game state at all times and no hidden or chance moves occur. The *payoff* is often ternary: a player either wins, draws, or loses.

A natural question arises: if these games are purely deterministic, why not just use a computer to solve them? The short answer is that although brute force is theoretically possible, the search space is typically enormous. For instance, consider a $5 \times 5 \times 5$ three-dimensional version of Tic-Tac-Toe: there are on the order of 3^{125} possible positions. Even a powerful computer cannot exhaustively enumerate such a tree in reasonable time.

This realization motivates the field of Combinatorial Game Theory (CGT), which blends classical mathematical methods (such as potential functions, the probabilistic method, or linear algebra) with combinatorial, graph-theoretic, and strategic arguments. In these notes, we introduce some classical examples and techniques used in analyzing certain combinatorial games.

2 Examples and Basic Techniques

We begin with several illustrative games and show some fundamental tools for tackling them.

2.1 Solitaire Army and Potential Functions

Solitaire Army Puzzle. In a solitaire army puzzle, you have a board (with holes) and a number of "soldiers" placed in some of the holes. A legal move consists of jumping one soldier over one or more adjacent soldiers (in a straight line) and removing the jumped soldiers. Each move reduces the total number of soldiers on the board by at least one. One variant takes place on the infinite integer lattice $\mathbb{Z} \times \mathbb{Z}$, with soldiers initially placed in the "negative half-plane" (say, all coordinates $x \leq 0$). Moves can only be made horizontally or vertically. We ask: How many soldiers are needed to push one soldier k steps forward (i.e., into $x = +k$)?

- $k = 1$: Clearly 2 soldiers suffice.
- $k = 2$: One can show that 4 soldiers suffice.

*Department of Mathematics, UCI. Email: asaff@uci.edu.

- $k = 3$: 8 soldiers suffice (a bit more tricky).
- $k = 4$: 20 soldiers suffice.
- $k = 5$: *Surprisingly, it is impossible!*

The impossibility for $k = 5$ was shown by John Conway in 1961, pioneering what we now call a *potential function* or *weight function* argument.

Theorem 2.1 (Conway, 1961)**.** *In the Solitaire Army played on the infinite integer lattice $\mathbb{Z} \times \mathbb{Z}$, where soldiers can only jump horizontally or vertically and start in the* negative half-plane *($x \leq 0$), it is **impossible** to push any soldier* five *steps forward (that is, to a coordinate $x = +5$).*

Sketch with Details. We prove that no matter how many soldiers you initially place in the region $\{(x, y) \mid x \leq 0\}$, you cannot end up with a soldier at $(5, y_0)$. The argument relies on assigning a suitable *weight function* $\omega : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_{>0}$ to the lattice points, then showing that:

1. any *legal* jump cannot increase the total weight of occupied point, and
2. The total weight of any position with a soldier pushed 5 steps forward is larger than the total weight of any possible initial position.

Clearly, these two properties imply that there is no way to push a soldier 5 steps forward.

Step 1. Defining the Weight Function.

We choose a real number $\alpha \in (0, 1)$ such that

$$\alpha \; + \; \alpha^2 \; = \; 1.$$

(One can solve this quadratic to get $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.618\ldots$, the reciprocal of the golden ratio.)

Assigning weights. We want to forbid placing a soldier at $\big(5, 0\big)$. Hence, we set $\omega(5, 0) = 1$. From there, if we move *down* one unit in the lattice, we multiply the weight by α, so $\omega(5, -1) = \alpha$. Another step down yields $\omega(5, -2) = \alpha^2$, and so on. Similarly, from $\big(5, i\big)$ we move horizontally to $\big(4, i\big)$ by multiplying by α. In essence, each horizontal or vertical step changes the weight by a factor of α.

The crucial property we require is:

$$\omega(A) \; + \; \omega(B) \; \geq \; \omega(C),$$

whenever A, B, C are three collinear consecutive points with B in the middle. Let us see why that holds:

- Suppose $\omega(A) = w$.
- Then B is one step from A (vertically or horizontally), so $\omega(B)$ is either αw or w/α.
- And C is one step from B (same line), so its weight is $\alpha^2 w$ or w/α^2, or $\alpha w/\alpha$, etc.

  A direct check with $\alpha + \alpha^2 = 1$ shows:

  $$w + \alpha w = w(1 + \alpha) \; = \; w(\alpha + \alpha^2 + \alpha) \; = \; w(\alpha^2 + 2\alpha) \; \geq \; w \cdot \alpha^2.$$

  Hence, $\omega(A) + \omega(B) \geq \omega(C)$.

Step 2. Why the Total Weight Never Increases with a Legal Jump.

A *legal* jump moves a soldier from point A over an adjacent soldier at B onto the point C, then removes the soldier that was at B. Hence, before the move, points A and B are occupied. After the move, only C is newly occupied (while A and B no longer hold soldiers). Therefore, the net change in total weight of occupied sites is:

$$\Delta = \omega(C) \; - \; [\,\omega(A) + \omega(B)\,].$$

Since $\omega(A) + \omega(B) \geq \omega(C)$, it follows $\Delta \leq 0$. Thus, each legal move can only *maintain* or *decrease* the sum of the occupied weights.

Step 3. Weight of the Negative Half-Plane Is At Most 1.

By construction, $\omega(5, 0) = 1$. We now show that the total weight of any finite configuration of soldiers placed entirely in $\{x \leq 0\}$ is ≤ 1. Indeed, as we go from $(5, 0)$ down and left, each step multiplies the weight by factors of α. One checks carefully (through a neat telescoping or summation argument) that row by row, column by column, the sum of the weights in the $\{x \leq 0\}$ portion does not exceed 1. In particular, *no single* point in the negative half-plane is assigned weight 0; everything contributes some positive amount, yet the entire half-plane adds up to at most 1.

Hence, if you start with finitely many soldiers in $x \leq 0$, the *initial total weight* is less than 1.

Step 4. No Soldier Can Reach $(5, 0)$.

If at some time a soldier does occupy $(5, 0)$, that point alone contributes weight 1. But also, there might be other soldiers on the board, contributing *more* weight, giving a total > 1. However, we argued that every jump is weight-non-increasing and the *initial* weight (all soldiers in $\{x \leq 0\}$) was < 1. This is a contradiction, so you cannot legally arrive at a position containing a soldier at $(5, 0)$. Equivalently, one cannot push any soldier a full five steps forward from the negative side.

This completes the proof that a jump of length 5 is impossible, no matter how many soldiers are placed in $\{x \leq 0\}$. □

The takeaway is that a well-chosen *potential* (or *weight*) function can show that some apparently feasible position is, in fact, impossible. This technique recurs in many places in combinatorics and combinatorial games.

2.2 Strategy Stealing

Strategy stealing is a classic argument to prove that in many impartial or partizan games, the first player (the one who moves first) cannot be at a disadvantage. Although it rarely yields an *explicit* winning strategy, it often shows at least a guaranteed draw or a guaranteed win for Player I.

Theorem 2.2 (Strategy Stealing). *In a two-player perfect-information game where extra moves cannot harm, the first player can force at least a draw.*

Sketch. Suppose, for contradiction, that the second player has a *winning* strategy. Then the first player can "steal" it. Concretely:

1. On her first turn, the first player makes an arbitrary move.
2. Then, from that point onward, she pretends she is the second player and follows the second player's winning strategy, ignoring the existence of her extra first move.

If following that strategy ever demands a move that was already taken by her extra turn, she can choose any other free move (no penalty arises because extra moves do not harm). This forces at least a draw, contradicting the hypothesis that the second player could have forced a win. □

Example: Positional Games. A *positional game* usually means the following:

- There is a *board* V, whose elements are called *cells*, *points*, or *positions*.

- There is a family $\mathcal{F}$ of *winning sets*, each a subset of V.
- Two players, I and II, alternate turns claiming free cells of V. If at any point a player fully occupies all cells of one of the winning sets, that player wins immediately.
- If neither player occupies a winning set by the time V is exhausted, the game is a draw.

Tic-Tac-Toe is a quintessential example. In such games, Theorem 2.2 guarantees that Player I can never lose (i.e., she can force at least a draw).

2.3 Clique Games and Ramsey Theory

Consider the *k-clique game on K_n*:

- The board is the edge set $E(K_n)$ of the complete graph on n vertices.
- Each player in turn claims an unclaimed edge of K_n.
- The winning sets are all K_k subgraphs, i.e., sets of $\binom{k}{2}$ edges that form a k-clique.

A fundamental result of Ramsey theory states:

Theorem 2.3 (Ramsey)**.** *For every k, there is an integer $R(k)$ such that any graph on $n \geq R(k)$ vertices must contain either a K_k or an independent set of size k in its complement.*

From this, it follows that if $n \geq R(k)$, then *no position can be a draw*: any complete coloring of $E(K_n)$ into two colors (say, belonging to Players I and II) inevitably produces a monochromatic K_k in one color or the other. Hence one player *must* occupy all edges of some K_k. By strategy stealing, the first player thus has a winning strategy. However, the explicit nature of that strategy is still a mystery for all $k \geq 5$:

Problem 2.4. *For $k \geq 5$, find an explicit winning strategy for the first player in the k-clique game on K_n (with $n \geq R(k)$).*

Problem 2.5. *Determine whether the infinite k-clique game on $E(K_{\mathbb{N}})$ is a draw or a first-player win.*

2.4 Hex and the Non-Existence of Draws

The game of *Hex* was invented independently by Piet Hein and John Nash in the 1940s. The board is an $n \times n$ rhombus of hexagonal cells. The two players, conventionally called Red and Blue, own opposite edges of the rhombus and alternate claiming a previously unoccupied cell in order to form a continuous path connecting their two edges. Nash proved:

Theorem 2.6. *Hex cannot end in a draw.* *In the game of Hex on an $n \times n$ rhombus of hexagonal cells, if all cells are claimed (Red or Blue) and no further moves remain, then one of the players necessarily has a winning chain connecting their opposing edges.*

Proof. We provide a constructive argument, often called the "no-draw" or "bridge-and-cut" proof. We will show that in *every* fully marked Hex board, there is either a Red-crossing or a Blue-crossing, hence no draw is possible.

1. Representing the Board via a Planar "Corner Graph."

Consider the Hex board of side length n. Each hexagon has six *corners*. We place a vertex of a planar graph G at each corner of every hexagon. Two vertices (corners) in G are joined by an edge whenever they lie on the same boundary segment of a single hex cell. Because each hex cell is a regular hexagon, it has 6 edges and 6 corners, so each cell contributes edges between its corners.

Additionally, we attach four *external* vertices:

$$U_1, \quad U_2 \quad U_3, \quad U_4 \quad \text{(for each corner of the rhombus).}$$

Each external vertex U_i is connected to a corner vertex.

2. Color Each Cell Red or Blue, and Create the "Bridging Graph."

Once the game ends, each hex cell is fully claimed by exactly one player: Red or Blue. We now form a subgraph $G' \subseteq G$ (often called the *bridging graph*) as follows:

1. For each corner vertex x in the "core" of G (i.e., not one of the four external vertices), look at the *two or three* distinct cells that meet at x.
   - If all these cells are of the *same color*, then x is *isolated* in G' (it has degree 0 there). In other words, x has no edges in G'.
   - Otherwise, there are cells of both colors meeting at x. We connect x with an edge in G' precisely along each boundary line that separates a Red cell from a Blue cell. (One corner can be incident to 2 or 3 cells, so typically x ends up with degree 2 in G', explained below.)
2. The four external vertices $\{U_1, U_2, U_3, U_4\}$ remain in G' and keep their edges to the corners (since each boundary line is between a Red-face side and a Blue-face side).

The edges of G' therefore lie exactly along the *common border* of a Red cell and a Blue cell; hence we say G' is built from "Red-Blue boundaries." Notice that each external vertex U_i is connected to a corner on the perimeter – specifically where a boundary might be shared by Red and Blue cells.

3. Degree Analysis: $\deg(x) \leq 2$**.**

Consider any corner vertex x in the interior (non-external):

- If x touches *three* cells of the same color, then x is isolated in G' (degree 0). No edges connect it to other corners in G', because no Red/Blue boundary crosses that corner.
- If x touches exactly *two* cells of color A (say Red) and one cell of color B (Blue), or vice versa, then exactly *two* edges of G' emanate from x, each corresponding to the boundary line with the cell of the *opposite* color. Thus x has degree 2 in G'.
- It is impossible for a corner to see three distinct cells of *two or more* different colors in more complicated ways: each hex corner is shared by at most 3 cells, so the only patterns are (3 same color) or (2 of one color, 1 of the other).

Hence, every interior vertex has degree 0 or 2 in G'. Because we also connect external vertices U_i each to exactly one corner, each external vertex has *degree 1* in G'.

4. Union of Paths, Cycles, and Isolated Vertices.

We now apply a simple fact: a finite graph whose vertices all have degree 0 or 2 is a disjoint union of *isolated vertices*, *simple cycles*, and *simple paths.*

Lemma 2.7. *Let H be a finite graph in which every vertex has degree* 0, 1*, or* 2*. Then each connected component of H is either:*

1. *an isolated vertex (degree* 0*),*
2. *a simple cycle, or*
3. *a simple path with two endpoints.*

Sketch of lemma. Pick a connected component of H. If it has no edges, it is an isolated vertex (degree 0). Otherwise, pick any vertex v in that component (degree ≥ 1). Following an edge from v to another vertex w, we must continue to a unique next vertex (since each has degree 2, so exactly two edges). We get a chain that either *closes* into a cycle or eventually terminates if we ever reach a vertex with degree 1. Because H is finite, the chain cannot continue indefinitely without repeating a vertex. Thus each connected component is a path or a cycle. □

By this lemma, G' is a disjoint union of isolated corners, cycles, and paths.

5. Paths Must Connect Two Distinct External Vertices.

Because each external vertex U_i is connected to the board and has degree 1, it must appear as an *endpoint* of some path in G' (it cannot be in a cycle or be isolated since its degree is not 2 or 0). Consequently, for each connected component containing U_i, we get a *path* whose two endpoints are external vertices. Therefore, in G' there must be a path connecting *some* U_i to *some* U_j.

A moment's thought now reveals that this path corresponds to a Red-crossing or a Blue-crossing path. Thus one of the two colors obtains a top-to-bottom or side-to-side chain.

Since no position can realize a disconnected boundary for *both* Red and Blue simultaneously (the bridging graph structure prevents that), exactly one color must have an unbroken path from one boundary side to the other. This *ensures a win for that color.*

Conclusion. Because our final G' reveals a path connecting the two edges of *some* player, Hex cannot end in a draw: exactly one of Red or Blue has a continuous chain across the board. This completes the proof that **Hex is draw-free**. □

Corollary 2.8. *By the* strategy-stealing *argument, the first player on an $n \times n$ Hex board has a guaranteed winning strategy (though finding an explicit construction is still an open problem for large n).*

2.5 Shannon's Switching Game

Shannon's Switching Game is played on a *multigraph* G:

- The board is $E(G)$, the set of edges of G (multiple edges allowed).
- Two players, called Maker and Breaker, alternate claiming edges, with Breaker going first.
- Maker attempts to build a spanning tree (i.e., claim edges so that every vertex is in one connected component).
- Breaker tries to prevent that.

Lehman's 1964 solution is beautiful and concise:

Theorem 2.9 (Lehman)**.** *Maker (as the second player) can force a spanning tree if and only if G contains* two *edge-disjoint spanning trees.*

Sketch. If G has two edge-disjoint spanning trees, then Maker's strategy is: whenever Breaker claims some edge that disconnects one of the two trees, Maker immediately claims an edge xy in the *other* tree that reconnects those components. Then, Maker can "contract" xy into a new vertex z to reduce G by one vertex each time, but it still has two spanning trees. By induction, Maker achieves a spanning tree in the final position.

Conversely, if Maker has a winning strategy, a *strategy-stealing* argument shows that Breaker (as if he were second) could also assemble a spanning tree. Hence G must simultaneously support two edge-disjoint spanning trees. □

2.6 Pairing Strategies

Consider a hypergraph $H = (V, \mathcal{F})$ describing a combinatorial game. A simple but powerful *pairing strategy* for the second player is possible if one can partition a subset of V into disjoint 2-element blocks (pairs) such that *every winning set in $\mathcal{F}$ contains at least one of these pairs.* Then, whenever the first player claims one element of a pair, the second player immediately claims the other, blocking that pair entirely and preventing a winning set from being fully claimed by Player I.

Definition 2.10. *A* pairing strategy *is a matching in V such that every set in $\mathcal{F}$ contains at least one of the matched pairs. Claiming the other element whenever the first is taken yields at least a draw for the pairing player.*

A classic sufficient condition for the existence of such a matching is given by Hall's theorem when the hyperedges are almost disjoint or each winning set is of size 2 in some bipartite representation.

Theorem 2.11 (Hales–Jewett, 1963; Pairing Strategy Draw)**.** *Suppose $\mathcal{F}$ is a family of subsets of V such that for every subfamily $\mathcal{H} \subseteq \mathcal{F}$, we have*

$$\left| \bigcup_{A \in \mathcal{H}} A \right| \;\geq\; 2|\mathcal{H}|.$$

Then the second player can force a pairing strategy draw (or the first player, by the same argument).

A basic corollary (via Hall's theorem) is:

Theorem 2.12 (Degree Condition for Pairing Strategy)**.** *Let $\mathcal{F}$ be an n-uniform hypergraph on V. Suppose that every element of V belongs to at most $n/2$ sets in $\mathcal{F}$. Then a pairing strategy draw exists.*

2.7 Erdős–Selfridge Criterion and the Method of Conditional Expectation

Another high-impact technique uses the *method of potential functions* or *conditional expectation* to show that the second player can force a *strong draw*—meaning the first player can never complete a winning set. Erdős and Selfridge showed the following global criterion:

Theorem 2.13 (Erdős–Selfridge, 1973)**.** *Let $\mathcal{F}$ be an n-uniform hypergraph on V. If*

$$|\mathcal{F}| \;+\; \max\{deg_{\mathcal{F}}(v)\} \;<\; 2^n,$$

then the second player has a strategy that prevents any set in $\mathcal{F}$ from being fully claimed by Player I (i.e., a strong draw).

Sketch. Number the sets $\mathcal{F} = \{A_1, \ldots, A_M\}$. In each turn, the first player claims an element, and then the second player responds, removing all sets that contain the chosen element (these become "dead"). We define a *danger function* or potential

$$D_i \;=\; \sum_{s \in S_i} 2^{-u_s},$$

where S_i is the set of "surviving" (still-alive) hyperedges at round i, and u_s is the number of unclaimed elements in s. A careful choice of the second player's move reduces D_i or at least ensures $D_{i+1} \leq D_i$. Initially,

$$D_1 \;\leq\; \frac{|\mathcal{F}| + \max\deg(\mathcal{F})}{2^n} \;<\; 1.$$

However, if Player I were to succeed in occupying some entire set A_j, we would have $D_{\text{final}} \geq 1$. Because our strategy ensures D_i never increases, this is impossible. Hence the second player forces a strong draw. □

Remark 2.14. *This result can be seen as a derandomization of the simpler probabilistic observation that if* $|\mathcal{F}| < 2^n$*, then a random coloring often has no monochromatic set* A_i *in one color. Erdős–Selfridge's method upgrades this existence into an actual* strategy*.*

Examples and Applications.

- 4^2 *and* 8^3 *Tic-Tac-Toe*:
$$10 + \max\deg = 10 + 3 = 13 \; < \; 2^4 = 16.$$
By Erdős–Selfridge, the second player can force a draw in 4×4 Tic-Tac-Toe. Similarly, with a more thorough count, the second player can also draw in $8 \times 8 \times 8$.

- *Maker's Connectivity Game on* K_n: One can show using a similar potential argument (or a variant) that Maker (as second player) can ensure building a spanning tree in the edge set of K_n by preventing certain cutsets from being claimed by Breaker, etc.

3 Biased Games

In many *unbiased* games (each player claims one cell/edge each turn), the advantage might be so heavily with Maker/Player I that it becomes interesting to *bias* the game: let Maker claim m elements per move and Breaker claim b elements per move. One prototypical example is the *Triangle Game on* K_n with a (m, b) bias. The question: does Maker still have a winning strategy to create a triangle?

A simpler model is the *Box Game* $Box(p, q; a_1, \ldots, a_n)$:

- The board consists of n disjoint sets ("boxes") of sizes a_i.
- On each turn, Maker claims p free elements while Breaker claims q free elements.
- Maker's goal is to fully occupy at least one box.

Chvátal and Erdős studied how p and q (and the sizes a_i) determine the outcome. One finds a phase transition in the ratio p/q, reminiscent of threshold phenomena. Using potential-like arguments or enumerations, one can pinpoint approximate boundaries for Maker's guaranteed success.

3.1 A Biased Version of Erdős–Selfridge (Beck's Theorem)

Beck extended Erdős–Selfridge to the (p, q) game. The rough statement is:

Theorem 3.1 (Beck, Biased Erdős–Selfridge)**.** *Let* p, q *be positive integers. If*

$$\sum_{A \in \mathcal{F}} (1 + q)^{-\frac{|A|}{p}} \; < \; \frac{1}{1 + q},$$

then in the (p, q) *game on* $\mathcal{F}$*, Breaker can prevent any* $A \in \mathcal{F}$ *from being fully claimed by Maker.*

Thus, even for heavily biased games, a suitable global criterion can guarantee a draw (or a Breaker's win, depending on the viewpoint).

4 Random (Mixed) Strategies

Even though Combinatorial Games are deterministic, sometimes a *random strategy* can help a player *guarantee* (with positive probability) a desired outcome, which implies (by determinacy) that a deterministic winning/drawing strategy exists.

Theorem 4.1 (A Minimum-Degree Game Example)**.** *For every $\epsilon > 0$, there is a constant C such that the following holds. Suppose G is a graph on n vertices with minimum degree $d \geq C \log n$. Then Maker can adopt a random strategy ensuring, with probability $1 - o(1)$, that the subgraph Maker claims has minimum degree at least $(1 - \epsilon)\frac{d}{3}$.*

Sketch. Whenever Breaker claims an edge uv, Maker flips a fair coin and randomly claims either an edge incident to u or to v. One applies Chernoff bounds and a union bound to show that, with high probability, Maker's final subgraph has the desired minimum degree. By determinacy, Maker thus also has a *deterministic* way to force $\delta(M) \geq (1 - \epsilon)\frac{d}{3}$. □

Random strategies also appear in analyzing linear or sublinear Maker-Breaker games, bounding threshold biases, and more.

5 Further Directions and Concluding Remarks

We have seen an assortment of classical ideas:

- **Potential/Weight Functions:** Used to show some outcomes are impossible (Conway's Army).
- **Strategy Stealing:** Ensures the first mover never does worse than draw (positional games, Hex).
- **Pairing/Matching Strategies:** Hall's theorem ensures a draw if each winning set intersects certain paired blocks.
- **Probabilistic/Expectation Methods:** Erdős–Selfridge and Beck's extension show how to systematically "derandomize" a draw strategy under certain size or neighborhood conditions.
- **Random/Probabilistic Strategies:** Maker can flip coins to ensure certain global properties, then convert to a deterministic solution.

Despite this progress, many natural games (like large clique games, many scoring Tic-Tac-Toe variations, or infinite-board games) remain *open* in key respects, with no known explicit winning strategies.

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